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# W-Strings on Group Manifolds

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**ABSTRACT**

We present a procedure for constructing actions describing propagation of W-strings on group manifolds by using the Hamiltonian canonical formalism and representations of W-algebras in terms of Kac-Moody currents. An explicit construction is given in the case of the $W_3$ string.

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W-string (or W-gravity) theories are higher spin generalizations of ordinary string theories, such that two-dimensional (2d) matter is not only coupled to 2d metric but also to a set of higher spin 2d gauge fields (for a review see [1]). Since ordinary string theory can be considered as a gauge theory based on the Virasoro algebra, one can analogously define a W-string theory as a gauge theory based on a W-algebra [2] (or any other higher spin conformally extended algebra [1]). Actions for a large class of W-string theories have been constructed so far [3-10]. These actions essentially describe a W-string propagating on a flat background. In this letter we would like to address the problem of constructing the action for a W-string propagating on a curved background by studying the special case of a group manifold.

We are going to use a general method for constructing gauge invariant actions, based on the Hamiltonian canonical formalism [9]. This method works if one knows a representation of the algebra of gauge symmetries in terms of the coordinates and canonically conjugate momenta. The basic idea is simple: given a set of canonical pairs $(p_i, q^i)$ together with the Hamiltonian $H_0(p,q)$ and constraints $G_\alpha(p,q)$ such that

$$\{G_\alpha, G_\beta\} = f_{\alpha\beta}{}^\gamma G_\gamma \quad , \tag{1}$$

$$\{G_\alpha, H_0\} = h_\alpha{}^\beta G_\beta \quad , \tag{2}$$

where $\{,\}$ is the Poisson bracket and (1) is the desired algebra of gauge symmetries, then the corresponding action is given by

$$S = \int dt \left( p_i \dot{q}^i - H_0 - \lambda^\alpha G_\alpha \right) \quad . \tag{3}$$

The parameter $t$ is the time and dot denotes time derivative. The Lagrange multipliers $\lambda^\alpha(t)$ play the role of the gauge fields associated with the gauge symmetries generated by $G_\alpha$. The indices $i, \alpha$ can take both the discrete and the continious values. Note that the coeficients $f_{\alpha\beta}{}^\gamma$ and $h_\alpha{}^\beta$ can be arbitrary functions of $p_i$ and $q^i$, and hence the algebra (1) is general enough to accomodate the case of the $W$ algebras, where the right-hand side of the Eq. (1) is a non-linear function of the generators. The action $S$ is invariant under the following gauge transformations

$$\begin{aligned} \delta p_i &= \epsilon^\alpha \{G_\alpha, p_i\} \\ \delta q^i &= \epsilon^\alpha \{G_\alpha, q^i\} \\ \delta\lambda^\alpha &= \dot{\epsilon}^\alpha - \lambda^\beta \epsilon^\gamma f_{\gamma\beta}{}^\alpha - \epsilon^\beta h_\beta{}^\alpha \quad . \end{aligned} \tag{4}$$

It is clear from the the transformation law for $\lambda^\alpha$ why they can be identified as gauge fields.

Since we want to describe propagation of a bosonic W-string on a curved background, the canonical coordinates will be a set of 2d scalar fields $\phi^a(\sigma,\tau)$, $a = 1,...,n$, where $\sigma$ is the string coordinate ($0 \leq \sigma \leq 2\pi$) and $\tau$ is the evolution parameter. $\phi^a$ are coordinates on an $n$-dimensional space-time manifold $M$, and we are going to study the special case when $M$ is a Lie group $G$. Let $\pi_a(\sigma,\tau)$ be the canonically conjugate momenta, satisfying

$$\{\phi^a(\sigma_1,\tau),\pi_b(\sigma_2,\tau)\} = \delta^a_b\delta(\sigma_1-\sigma_2) \quad . \tag{5}$$

In order to construct the desired action, we need a canonical representation of the corresponding W-algebra. This can be obtained from the canonical analysis of the Wess-Zumino-Novikov-Witten (WZNW) action and the fact that the generators of a W-algebra can be obtained as traces of products of the Kac-Moody currents [7,11,12].

The WZNW action can be written as

$$S_2 = \kappa\int d^2\sigma\left(-\tfrac{1}{2}\sqrt{-g}g^{\mu\nu}H_{ab}(\phi) + \epsilon^{\mu\nu}\mathcal{T}_{ab}(\phi)\right)\partial_\mu\phi^a\partial_\nu\phi^b \quad , \tag{6}$$

where $g_{\mu\nu}$ is a 2d metric, $\epsilon^{\mu\nu}$ is an antisymmetric 2d tensor density, $\partial_\mu = (\partial_0,\partial_1) = (\partial_\tau,\partial_\sigma)$, $H_{ab}$ is the metric on the manifold $G$, while $\mathcal{T}_{ab}$ is an antisymmetric tensor field. These tensors can be defined through left/right invariant Maurer-Cartan one-forms on $G$

$$v_+ = g^{-1}dg \quad , \quad v_- = gdg^{-1} = -dgg^{-1} \quad , \quad g \in G \tag{7}$$

such that

$$H_{ab} = Tr(E_{Aa},E_{Ab}) = \gamma_{\alpha\beta}E^\alpha_{Aa}E^\beta_{Ab} \quad , \quad v_A = d\phi^a E^\alpha_{Aa}\,t_\alpha \quad , \tag{8}$$

and

$$Tr(v_+^3) = -6d\mathcal{T} \quad , \quad \mathcal{T} = \tfrac{1}{2}\mathcal{T}_{ab}\,d\phi^a\wedge d\phi^b \quad , \tag{9}$$

where $A = \pm$, $E$'s are veilbeins on $G$, $t_\alpha$ are the generators of the Lie algebra of $G$, $[t_\alpha,t_\beta] = f_{\alpha\beta}{}^\gamma t_\gamma$, and $\gamma_{\alpha\beta} = f_{\alpha\gamma}{}^\delta f_{\beta\delta}{}^\gamma$ is the group metric.

The canonical form of the action (6) can be written as

$$S_2 = \int_{\tau_1}^{\tau_2}d\tau\int_0^{2\pi}d\sigma\left(\pi_a\dot\phi^a - h^A T_A\right) \quad , \tag{10}$$

where the constraints $T_A$ are given by

$$T_A = \frac{1}{4\kappa}Tr(J_A^2) = \frac{1}{4\kappa}\gamma^{\alpha\beta}J_{A\alpha}J_{A\beta} \quad , \tag{11}$$

$$J_{A\alpha} = -E^a_{A\alpha}(\pi_a + 2\kappa\mathcal{T}_{ab}\phi'^b) - (-1)^A\kappa E_{Aa\alpha}\phi'^a \quad , \tag{12}$$

where primes stand for the $\sigma$ derivatives. The constraints $T_A$ are $++$ and $--$ components of the energy-momentum tensor ($A^{\pm} = A^0 \pm A^1$), and $T_A$ satisfy the Virasoro algebra

$$\{T_{\pm}(\sigma_1), T_{\pm}(\sigma_2)\} = \mp\delta'(\sigma_1 - \sigma_2)(T_{\pm}(\sigma_1) + T_{\pm}(\sigma_2)) \tag{13}$$

under the Poisson brackets (5). The currents $J_{A\alpha}$ satisfy the Kac-Moody algebra

$$\{J_{\pm\alpha}(\sigma_1), J_{\pm\beta}(\sigma_2)\} = f_{\alpha\beta}{}^{\gamma} J_{\pm\gamma}(\sigma_1)\delta(\sigma_1 - \sigma_2) \pm 2\kappa\gamma_{\alpha\beta}\delta'(\sigma_1 - \sigma_2) \quad . \tag{14}$$

As usual, the plus and minus currents have vanishing Possion brackets.

Formulas (11) and (12) are the basis for building a canonical representation of a $W$ algebra, since we can write

$$W_{As} = \tfrac{1}{s} d^{\alpha_1 \cdots \alpha_s} J_{A\alpha_1} \cdots J_{A\alpha_s} \quad (s = 2, ..., N) \quad . \tag{15}$$

The coeficients $d^{\alpha_1 \ldots \alpha_s}$ will be determined from the requirement that the Poisson bracket algebra of $W$'s closes (or equivalently, $W$'s are first class constraints). The results of [7,11,12] imply that the general relation (15) can be simplified to

$$W_{As} = \frac{1}{2\kappa s} Tr({J_A}^s) \quad , \tag{16}$$

where $J_A = J_A^{\alpha} t_{\alpha}$. In the case of the $W_3$ algebra we have [12]

$$W_{A3} = \frac{1}{6\kappa} Tr(J_A^3) \quad , \tag{17}$$

so that $d_{\alpha\beta\gamma} = \frac{1}{4\kappa} Tr(t_{\alpha}\{t_{\beta}, t_{\gamma}\})$. One can check that $T$ and $W$ given by (11) and (17) form a classical $W_3$ algebra

$$\begin{aligned} \{T_{\pm}(\sigma_1), T_{\pm}(\sigma_2)\} &= \mp\delta'(\sigma_1 - \sigma_2)(T_{\pm}(\sigma_1) + T_{\pm}(\sigma_2)) \\ \{T_{\pm}(\sigma_1), W_{\pm}(\sigma_2)\} &= \mp\delta'(\sigma_1 - \sigma_2)(W_{\pm}(\sigma_1) + 2W_{\pm}(\sigma_2)) \\ \{W_{\pm}(\sigma_1), W_{\pm}(\sigma_2)\} &= \mp 2c\delta'(\sigma_1 - \sigma_2)({T^2}_{\pm}(\sigma_1) + {T^2}_{\pm}(\sigma_2)) \quad , \end{aligned} \tag{18}$$

and all other Poisson brackets are zero. Here $W = W_3$, while $c = c_1 - \frac{1}{n}$, where $c_1$ is a constant defined by the relation

$$Tr(J^4) = c_1 (Tr J^2)^2 \quad . \tag{19}$$

The relation (19) is valid for $G = A_l, B_l, C_l$, $l \leq 2$, since for the groups of rank $l > 2$, $Tr(J^4)$ is an independent Casimir invariant [12].

The 2d diffeomorphism invariance requires $H_0 = 0$. Otherwise, the wavefunctional $\Psi[\phi]$ would depend explicitely on the unphysical parameter $\tau$, since $i\frac{\partial}{\partial\tau}\Psi = \hat{H}_0\Psi$. Then according to the Eq. (3) the gauge invariant action is simply

$$S_N = \int_{\tau_1}^{\tau_2} d\tau \int_0^{2\pi} d\sigma \left( \pi_a \dot{\phi}^a - h^A T_A - \sum_{s=3}^{N} b^A{}_s W_{As} \right) \quad , \tag{20}$$

where $h$ and $b$ are the lagrange multipliers, which are also the gauge fields corresponding to the $W$-symmetries. The gauge transformation laws can be determined from the Eq. (4). In the $W_3$ case they become

$$\begin{aligned} \delta\pi_a = & \left( \epsilon^A \frac{\gamma^{\alpha\beta}}{2\kappa} + \xi^A d^{\alpha\beta\gamma} J_{A\gamma} \right) J_{A\alpha} \frac{\partial J_{A\beta}}{\partial \phi^a} \\ & + \kappa \left[ \left( \epsilon^A \frac{H^{bc}}{2\kappa} + \xi^A D_A{}^{bcd} J_{Ad} \right) J_{Ab} (2\mathcal{T}_{ca} + (-1)^A H_{ca}) \right]' , \end{aligned} \tag{21.a}$$

$$\delta\phi^a = \frac{\epsilon^A}{2\kappa} J_A{}^a + \xi^A D_A{}^a{}_{bc} J_A{}^b J_A{}^c \quad , \tag{21.b}$$

$$\delta h^A = \dot{\epsilon}^A - (-1)^A [h^A(\epsilon^A)' - (h^A)'\epsilon^A] + 2c(-1)^A [\xi^A (b^A)' - (\xi^A)' b^A] T_A \quad , \tag{21.c}$$

$$\delta b^A = \dot{\xi}^A + (-1)^A \left[ 2(h^A)'\xi^A - h^A(\xi^A)' - 2b^A(\epsilon^A)' + (b^A)'\epsilon^A \right] \quad , \tag{21.d}$$

where $\epsilon^A$ are the parameters of the $T_A$ transformations, $\xi^A$ are the parameters of the $W_A$ transformations, while

$$D_{Aabc}(\phi) = d_{\alpha\beta\gamma} E^\alpha_{Aa}(\phi) E^\beta_{Ab}(\phi) E^\gamma_{Ac}(\phi) \quad , \quad J_{Aa} = E^\alpha_{Aa} J_{A\alpha} \quad . \tag{22}$$

In all equations we use Einstein's summation convention, i.e. summation is performed only if the up and down index are the same.

In order to find a geometrical interpretation of the action (20) we need to know its second order form. It can be obtained by replacing the momenta $\pi_a$ in (20) by their expressions in terms of $\phi^a$. These expressions can be obtained from the equation of motion

$$\frac{\delta S_N}{\delta \pi_a} = 0 \quad . \tag{23}$$

In the $W_3$ case one gets

$$\dot{\phi}^a + \frac{h^A}{2\kappa} J^a_A + b^A D_A{}^a{}_{bc} J^b_A J^c_A = 0 \quad . \tag{24}$$

This is a quadratic equation in $\pi_a$, and therefore the second order form of the Lagrangian density will be a non-polynomial function of $\partial_\mu \phi$, $h$ and $b$, which can be written as an infinite power series in those variables. There is a complete analogy with the flat background (or Abelian $G$) case [9], where

$$J^a_\pm = -H^{ab}\pi_b - 2\kappa \mathcal{T}^a{}_b \phi'^b \mp \kappa\phi'^a \to -\pi_a \mp \kappa\phi'^a \quad . \tag{25}$$

Note that in the case of an arbitrary background $H_{ab}$, the expression (25) for $J$ (or equivalently Eq. (12)) is not useful for building the generators of a $W$ algebra since then the $J$'s do not satisfy the Poisson bracket Kac-Moody algebra (14).

As a preparation for the $W_3$ case, we first study the second order action obtained from the first order action (10) in the $W_2$ case. One can show that after the elimination of the momenta one obtains the covariant form of the WZNW action (6), after the following identifications

$$\tilde{g}^{00} = \frac{2}{h^+ + h^-} \quad , \quad \tilde{g}^{01} = \frac{h^- - h^+}{h^+ + h^-} \quad , \quad \tilde{g}^{11} = -\frac{2h^+h^-}{h^+ + h^-} \quad , \tag{26}$$

where $\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$. The covariant form of the 2d diffeomorphism transformations can be obtained from the Eq. (21.b), by rewritting it as

$$\delta\phi^a = \frac{\epsilon^A}{2\kappa} J_A{}^a = -\frac{\epsilon^A}{\sqrt{h^+ + h^-}} \tilde{e}_A{}^\mu \partial_\mu \phi^a = \epsilon^\mu \partial_\mu \phi^a \quad , \tag{27}$$

where

$$\tilde{e}_A{}^\mu = \frac{1}{\sqrt{h^+ + h^-}} \begin{pmatrix} 1 & h^- \\ 1 & -h^+ \end{pmatrix} \quad . \tag{28}$$

Eq. (21.c) can be rewritten as

$$\delta\tilde{g}^{\mu\nu} = -\partial_\rho(\epsilon^\rho \tilde{g}^{\mu\nu}) + \partial_\rho \epsilon^{(\mu|} \tilde{g}^{|\nu)\rho} \quad , \tag{29}$$

which is the diffeomorphism transformation of a densitized metric generated by the parameter $\epsilon^\mu$. The metric $g^{\mu\nu}$ can be written as

$$g^{\mu\nu} = \frac{1}{\sqrt{-g}(h^+ + h^-)} \begin{pmatrix} 2 & h^- - h^+ \\ h^- - h^+ & -2h^+h^- \end{pmatrix} = e_+{}^{(\mu|} e_-{}^{|\nu)} \quad , \tag{30}$$

where $e_A{}^\mu = (-g)^{-\frac{1}{4}} \tilde{e}_A{}^\mu$ are the zweibeins. Note that $\sqrt{-g}$ is undetermined, because the action (6) is independent of $\sqrt{-g}$ due to the Weyl symmetry

$$\delta g^{\mu\nu} = \omega g^{\mu\nu} \quad . \tag{31}$$

Also note that the relations (26,28,30) are essentialy the same as in the flat background case [9].

In the $W_3$ case we have from the Eq. (24)

$$\begin{aligned} \pi_a + 2\kappa \mathcal{T}_{ab} \phi'^b &= \kappa H_{ab}\, \tilde{g}^{0\mu} \partial_\mu \phi^b + \delta\Pi_a \\ \delta\Pi_a &= \frac{b^A}{h^+ + h^-} D_{Aabc} J_A^b J_A^c \quad , \end{aligned} \tag{32}$$

where $\tilde{g}^{0\mu}$ is given by the Eq. (26). Then the action (20) takes the following form

$$S_3 = \int d^2\sigma \left( \mathcal{L}_2 - \frac{h^+ + h^-}{4\kappa} \delta\Pi^a \delta\Pi_a - \frac{b^A}{3} D_{Aabc} J_A^a J_A^b J_A^c \right) \quad , \tag{33}$$

where $\mathcal{L}_2$ is the Lagrangian density of the WZWN action (6). Note that the Eq. (24) can be rewritten as

$$J_A^a = -\frac{2\kappa}{\sqrt{h^+ + h^-}} \partial_A \phi^a - \frac{b^A}{h^+ + h^-} D_A{}^a{}_{bc} J_A^b J_A^c \quad . \tag{34}$$

Eq. (34) can be used to obtain a power series expansion of $J_A$ in terms of $\partial_\pm \phi$, $h$ and $b$, which can be inserted into Eq. (33) to give the corresponding power series expansion of the action. Up to the first order in $b$ the Lagrange desity can be written as

$$\mathcal{L}_3 = \mathcal{L}_2 - b^{ABC} D_{+abc} \partial_A \phi^a \partial_B \phi^b \partial_C \phi^c + O(b^2) \quad , \tag{35}$$

where the only nonzero components of $b^{ABC}$ are

$$b^{\pm\pm\pm} = \pm \frac{4\kappa^3}{3} \frac{b^\pm}{(h^+ + h^-)^{\frac{3}{2}}} \quad , \tag{36}$$

and we used the property $D_{+abc} = -D_{-abc}$.

It is clear that the above procedure will give the following form of the second order covariant Lagrange density

$$\begin{aligned} \mathcal{L} = \mathcal{L}_2 &+ \tilde{b}^{\mu\nu\rho} D_{abc}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b \partial_\rho \phi^c \\ &+ \tilde{c}^{\mu\nu\rho\sigma} D_{ab}{}^e(\phi) D_{ecd}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b \partial_\rho \phi^c \partial_\sigma \phi^d + \cdots \quad , \end{aligned} \tag{37}$$

which for an Abelian $G$ reduces to the flat-space case [9]. The objects $\tilde{g}$, $\tilde{b}$, $\tilde{c}$, ... , must transform as tensor densities since the action is invariant under the infinitesimal diffeomorphisms

$$\delta\phi^a = \frac{\epsilon^A}{2\kappa} J_A^a = \epsilon^\mu \partial_\mu \phi^a \quad , \quad \epsilon^\mu = f^\mu(\epsilon^\pm, h^\pm, b^\pm, \phi^a, \partial_\mu \phi^a) \quad . \tag{38}$$

Besides the diffeomorphism invariance, the generalized Weyl symmetry [1] is also obscured. Heuristically it is there by construction, since we used only four independent gauge fields $h^\pm$ and $B^\pm$. The fields $\tilde{g}$, $\tilde{b}$, $\tilde{c}$, ... in the Eq. (37) are functions of $h$ and $b$, and one can check order by order in $\partial\phi$ that

$$\tilde{g}_{\mu\nu} \tilde{b}^{\mu\nu\rho} = 0 \quad , \quad \tilde{c}^{\mu\nu\rho\sigma} = \tilde{g}_{\tau\epsilon} \tilde{b}^{\mu\nu\tau} \tilde{b}^{\epsilon\rho\sigma} \quad , \tag{39}$$

and so on, which is the covariant form of the generalized Weyl symmetry.

In conclussion we can say that the propagation of the bosonic $W_3$ string on a curved background is described by a non-polynomial action whose Lagrange density is given by the Eq. (37). This action is of the similar form as the action in the flat background case [9], and the only difference is that the $d_{\alpha\beta\gamma}$ coeficients become functions of the fields $\phi^a$ via the Eq. (22). It remains to be explored how to generalize the transformation laws (21.a-b) to the case of an arbitrary background $H_{ab}(\phi)$.

Note that for a realistic W-string theory the group $G$ has to be non-compact. When $G$ is compact, the space-time metric $H_{ab}$ is of the Euclidian signature, and moreover, there are no propagating degrees of freedom classically, since the $T_A$ constraints imply

$$J^a_\pm = 0 \to \quad \pi_a = 0\ ,\ \phi'^a = 0 \quad . \tag{40}$$

In the quantum case one can get propagating degrees of freedom due to the anomalies which will appear in the $W$ algebra (see [13,14] for the $W_2$ case). Still, the compact $G$ construction can be relevant if the spacetime manifold is of the type $M^d \times G$ where $M^d$ is the $d$-dimensional Minkowski spacetime. However, one has to keep in mind that due to the nonlinearity of the $W$ algebra (except in the $W_2$ and $W_\infty$ case) one cannot construct a representation for $M^d \times G$ by adding the representations for $M^d$ and $G$.

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